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## A Note on the Linearization Around an Equilibrium

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**Abstract.** In this note conditions are derived under which the linearization of a system described by the set of behavioral differential equations  $f(w, w^{(1)}, \dots, w^{(L)}) = 0$  around the equilibrium point  $w^* = 0$  is, indeed,  $\frac{\partial f}{\partial w_0}(0) \Delta + \frac{\partial f}{\partial w_1}(0) \Delta^{(1)} + \dots + \frac{\partial f}{\partial w_L}(0) \Delta^{(L)} = 0$ .

### 1. INTRODUCTION

Linearization, the *raison d'être* of linear systems, is firmly established for systems described by state equations

$$\dot{x} = F(x, u), \quad y = H(x, u).$$

If  $0 = F(0, 0)$  and  $0 = H(0, 0)$ , then the linearized system becomes

$$\begin{aligned} \dot{\Delta}_x &= \frac{\partial F}{\partial x}(0, 0) \Delta_x + \frac{\partial F}{\partial u}(0, 0) \Delta_u, \\ \Delta_y &= \frac{\partial H}{\partial x}(0, 0) \Delta_x + \frac{\partial H}{\partial u}(0, 0) \Delta_u. \end{aligned}$$

Under suitable smoothness conditions, it is readily established that this linear system defines, indeed, the linear part of the map  $(x(0), u) \rightarrow y$  defined by the nonlinear differential equation.

However, more often than not, models derived from first principles will come to us in the form of high order differential equations, for example as behavioral equations of the type

$$g(y, y^{(1)}, \dots, y^{(L)}, u, u^{(1)}, \dots, u^{(L)}) = 0.$$

The problem of writing state equations for such systems is usually studied using differential algebraic techniques. In this paper we will study the problem of approximating this system. If  $g(0, 0, \dots, 0, 0, 0, \dots, 0) = 0$ , it is natural to look for a linear approximation.

In order to linearize around this equilibrium, we can do one of the two things. First, find a state representation and linearize. However, this is much easier said than done, a smooth state representation may not even exist; if it does, it may be impossible to find it [1,2], worse yet, we may not need it. A second approach is to linearize with the audacity of the classical applied mathematician: simply expand the right hand side in a Taylor series and keep the linear term. This is, of course, much easier to do, but *is it justified?* This is the problem addressed in this note. The results, by the way, are entirely as expected.

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## 2. TECHNIQUE

We will study the linearization of a dynamical system described by a set of (high order) differential or difference equations. We will describe the setting in the case of differential equations. Let  $f : (\mathbf{R}^q)^{L+1} \rightarrow \mathbf{R}^q$ , assume that it is  $C^\infty$ , and consider the dynamical system described by the behavioral differential equations

$$f(w, w^{(1)}, \dots, w^{(L)}) = 0. \quad (\text{NL})$$

Suppose that  $w^* \in \mathbf{R}^q$  (we will take  $w^* = 0$ ) is an *equilibrium*, that is  $f(0, 0, \dots, 0) = 0$ . Linearizing (NL) around this operating point yields the *linearized* differential equations

$$\dot{f}_0 \Delta + \dot{f}_1 \Delta^{(1)} + \dots + \dot{f}_L \Delta^{(L)} = 0, \quad (\text{L})$$

where  $\Delta : \mathbf{R} \rightarrow \mathbf{R}^q$ ,  $\dot{f}_0 := \frac{\partial f}{\partial w_0}(0, 0, \dots, 0), \dots, \dot{f}_L = \frac{\partial f}{\partial w_L}(0, 0, \dots, 0)$  are the real  $(g \times q)$  matrices obtained by taking the first partials of the map  $(w_0, w_1, \dots, w_L) \rightarrow f(w_0, w_1, \dots, w_L)$ .

The question arises when and, if so, in what sense, (L) describes approximately (NL) in the neighbourhood of  $w^*$ . We will see that a sufficient condition for (L) to be a linearization of (NL) is the following: Let  $w = (w^1, w^2, \dots, w^g, w^{g+1}, \dots, w^q)$ ,  $f = (f^1, f^2, \dots, f^g)$  denote the components of  $w$  and  $f$ . Assume that there exist non negative integers  $L_0, L_1, \dots, L_g$ , such that locally around 0, the nonlinear equations (NL) may be solved as

$$\begin{bmatrix} w^{1(L_1)} \\ w^{2(L_2)} \\ \vdots \\ w^{g(L_g)} \end{bmatrix} = r(w, w^{(1)}, \dots, w^{(L)}),$$

with no derivatives of  $w^i$  of order larger than  $(L_i - 1)$ , for  $i = 1, 2, \dots, g$ , on the right hand side. This condition, while representative, is only necessary, and our results are a bit stronger. Note that the order of the derivatives appearing in the variables  $w^{g+1}, \dots, w^q$  are arbitrary. In other words, the variables  $w^1, w^2, \dots, w^g$  need not play the role of outputs in the conventional meaning of the term.

Examples:

1.  $y^{(2)} + u^{(1)}y^{(1)} + \sin y = u$  is linearized by  $\Delta_y^{(2)} + \Delta_y = \Delta_u$ .
2.  $(y^{(2)})^2 + y^{(1)} + \sin y = u^{(1)}$  is linearized by  $\Delta_y^{(1)} + \Delta_y = \Delta_u^{(1)}$ .
3. It is not clear from our results what the linearization of  $(y^{(2)})^2 + u^{(1)}y^{(1)} + \sin y = u$  is.

## 3. FORMALISM

*Ça va sans dire* that we will consider the framework for discussing dynamical systems introduced in [3]. A *dynamical system* is a triple  $(T, W, B)$  with  $T \subset \mathbf{R}$  the *time-axis*,  $W$  the *signal space*, and  $B \subseteq W^T$  the *behavior*. We will take  $T = \mathbf{R}$  and  $W = \mathbf{R}^q$ . The dynamical system  $(\mathbf{R}, \mathbf{R}^q, B)$  is said to be *time-invariant* if  $\sigma^t B = B$  for all  $t \in \mathbf{R}$ , where  $\sigma^t$  denotes the *t-shift*. It is said to be *linear* if  $B$  is a linear subspace of  $(\mathbf{R}^q)^{\mathbf{R}}$ .

The equation (NL) provide an example of a dynamical system described by *behavioral differential equations*. Define its behavior by

$$B_{NL} := \{w \in C^\infty(\mathbf{R}; \mathbf{R}^q) \mid (\text{NL}) \text{ is satisfied}\}.$$

Clearly  $\Sigma_{NL} := (\mathbf{R}, \mathbf{R}^q, B_{NL})$  is time-invariant. Defining  $B_L$  analogously, from the equations (L), yields the dynamical system  $\Sigma_L := (\mathbf{R}, \mathbf{R}^q, B_L)$ , which is clearly linear and time-invariant.

Our notion of linearization depends on the notion of tangency. Let  $\Sigma_i = (\mathbf{R}, \mathbf{R}^q, B_i)$ ,  $i = 1, 2$ , be two time-invariant dynamical systems with  $B_i \subseteq C^\infty(\mathbf{R}; \mathbf{R}^q)$ . We will say that

$\Sigma_1$  and  $\Sigma_2$  are *tangent at 0*, if (i)  $0 \in B_i$ ,  $i = 1, 2$ , if (ii) for all sequences  $w_k \in B_1$ ,  $k \in N$ , such that  $w_k \xrightarrow[k \rightarrow \infty]{} 0$  (in the  $C^\infty$ -topology), there exists a sequence  $\tilde{w}_k \in B_2$ , such that for some  $N \in \mathbb{Z}_+$  and all  $T \geq 0$ , there holds

$$\frac{\|w_k - \tilde{w}_k\|_{C^N([-T, T]; \mathbb{R}^q)}}{\|w_k\|_{C^N([-T, T]; \mathbb{R}^q)}} \xrightarrow[k \rightarrow \infty]{} 0,$$

(where  $\|f\|_{C^N([-T, T]; \mathbb{R}^q)} := \max_{0 \leq l \leq N} \sup_{t \in [-T, T]} \|f^{(l)}(t)\|$ ),

and, finally, if (iii) the analogue of (ii) holds with the roles of  $B_1$  and  $B_2$  reversed.

If  $\Sigma_1$  and  $\Sigma_2$  are tangent at 0 and if  $\Sigma_2$  is linear, then we will call  $\Sigma_2$  the *linearization of  $\Sigma_1$* . This makes precise what it means that (L) defines the linearization of (NL). It is easy to see that the linearization, if it exists, is unique.

Note that the linearized equation (L) deduced from (NL) depend very much on the behavioral equation used to represent  $B_{NL}$ . The representation (NL) is far from unique, however. (In order to see this, replace  $f = (f^1, f^2, \dots, f^g)$  by  $((f^1)^2, (f^2)^2, \dots, (f^g)^2)$ . This does not change the behavior, but the linearized equations will become trivial!) This lack of uniqueness can be alleviated to some extent by considering  $f$  as defining a variety in  $\mathbb{R}^{q(L+1)}$ .

#### 4. RESULT

Consider (L). Introduce the polynomial matrix  $R(s) = f'_0 + f'_1 s + \dots + f'_L s^L \in \mathbb{R}^{g \times q}[s]$ . Let  $g'$  be the rank of  $R(s)$ . Then there will exist a unimodular polynomial matrix  $U(s) =$

$U_0 + U_1 s + \dots + U_L s^L$  such that  $UR$  is of the form  $\begin{bmatrix} \tilde{R} \\ \dots \\ \mathbf{0} \end{bmatrix}$  with  $\tilde{R}(s) \in \mathbb{R}^{g' \times q}[s]$  row proper,

that is,

$$\tilde{R}(s) = \begin{bmatrix} r_{1,L_1} s^{L_1} + r_{1,L_1-1} s^{L_1-1} + \dots + r_{1,0} \\ r_{2,L_2} s^{L_2} + r_{2,L_2-1} s^{L_2-1} + \dots + r_{2,0} \\ \vdots \\ r_{g',L_{g'}} s^{L_{g'}} + r_{g',L_{g'}-1} s^{L_{g'}-1} + \dots + r_{g',0} \end{bmatrix}$$

is such that  $\tilde{R}_{\text{lead}} := \begin{bmatrix} r_{1,L_1} \\ r_{2,L_2} \\ \vdots \\ r_{g',L_{g'}} \end{bmatrix}$  is of full row rank. Now assume that the distinct integers

$1 \leq q_1, q_2, \dots, q_{g'} \leq q$  are such that the corresponding columns of  $\tilde{R}_{\text{lead}}$  are independent.

Now return to the differential equation (NL):  $f(w, w^{(1)}, \dots, w^{(L)}) = 0$ . For simplicity, assume that  $f$  is defined as a map of  $(\mathbb{R}^q)^{\mathbb{Z}^+}$  to  $\mathbb{R}^g$ , but that it depends only on a finite number of arguments. We will now define, for each polynomial matrix  $F(s) \in \mathbb{R}^{g \times q}[s]$ , a new map  $F \diamond f$  as follows. For  $F(s) = F_0$ , define  $(F \diamond f)(w_0, w_1, \dots) := F_0 f(w_0, w_1, \dots)$ , for  $F(s) = s$ , define  $(F \diamond f)(w_0, w_1, \dots) := \frac{\partial f}{\partial w_0}(w_0, w_1, \dots) w_1 + \frac{\partial f}{\partial w_1}(w_0, w_1, \dots) w_2 + \dots + \frac{\partial f}{\partial w_k}(w_0, w_1, \dots) w_{k+1} + \dots$ , for  $F(s) = s^k$ , define  $(s^k \diamond f) = s \diamond (s^{k-1} \diamond f)$ , for  $F(s) = F_0 + F_1 s + \dots + F_n s^n$ , define

$$(F \diamond f) = (F_0 \diamond f + F_1 \diamond (s \diamond f) + \dots + F_n \diamond (s^n \diamond f)).$$

Consider now the differential equation

$$\tilde{f}(w, w^{(1)}, \dots, w^{(L)}) = 0, \quad (\widetilde{\text{NL}})$$

with  $\tilde{f} = U \diamond f$ . (It is easy to see that the linearization of  $(\widetilde{\text{NL}})$  around  $w^* := 0$  is—if it exists—described by

$$\tilde{R}\left(\frac{d}{dt}\right)w = 0,$$

with  $UR = \begin{bmatrix} \tilde{R} \\ \dots \\ \mathbf{0} \end{bmatrix}$ .) Note that  $\tilde{f} : (\mathbf{R}^q)^{L'} \rightarrow \mathbf{R}^q$  and that  $L'$  does not need to be larger than  $L$  plus the degree of  $U(s)$ .

Now assume that the map  $\tilde{f} = (\tilde{f}^1, \tilde{f}^2, \dots, \tilde{f}^g)$  satisfies the following conditions involving the integers  $g', L_1, L_2, \dots, L_{g'}$ , and  $q_1, q_2, \dots, q_{g'}$  defined earlier on:

- (i) either  $g' = g$  or  $\tilde{f}^{g'+1} = \dots = \tilde{f}^g = 0$ ;
- (ii) the  $(g' \times g')$  matrix with  $(i, j)$ -th element  $\frac{\partial \tilde{f}^i}{\partial w_{L_j}^{q_j}}(0, 0, \dots, 0)$  is nonsingular;
- (iii)  $\tilde{f}^i(w_0, w_1, \dots)$  does not depend on  $w_k^{q_j}$ .

Equivalently for (ii) and (iii), assume that the equation  $(\widetilde{\text{NL}})$  can, locally around 0, be solved as

$$\begin{bmatrix} w^{q_1(L_1)} \\ w^{q_2(L_2)} \\ \vdots \\ w^{q_{g'}(L_{g'})} \end{bmatrix} = \tilde{r}(w, w^{(1)}, \dots, w^{(L)}),$$

with only derivatives in  $w^j$  up to order  $L_j - 1$ , for  $1 \leq j \leq g'$ , on the right hand side.

**THEOREM.** Let  $R(s) = f'_0 + f'_1 s + \dots + f'_L s^L$ , and  $U(s)$  be unimodular and such that  $U(s)R(s) = \begin{bmatrix} \tilde{R}(s) \\ \dots \\ \mathbf{0} \end{bmatrix}$  with  $\tilde{R}$  row proper. Assume that there exist distinct integers

$1 \leq q_1, q_2, \dots, q_{g'} \leq q$ , satisfying the relevant conditions for  $\tilde{R}$  and that  $\tilde{f} = U \diamond f$  satisfies conditions (i), (ii) and (iii) above. Then the system with behavioral equations (L) defines a linearization of the system with behavioral equations (NL).

**PROOF:** We will only give a global outline. The proof starts from the following observation:  $w \in C^\infty$  satisfies  $f(w, w^{(1)}, \dots, w^{(L)}) = 0$  if and only if it satisfies  $U(\frac{d}{dt})f(w, w^{(1)}, \dots) = 0$ . This latter equation is precisely  $\tilde{f}(w, w^{(1)}, \dots) = 0$  with  $\tilde{f} = U \diamond f$ . Similarly,  $R(\frac{d}{dt})w = 0$  if and only if  $\tilde{R}(\frac{d}{dt})w = 0$ . This shows that it suffices to prove the claim made in the theorem for the system discussed in Section 2. The linearization in this case can be proven by associating with each  $w \in B_L$  a  $\tilde{w} \in B_{NL}$  and vice versa.

Let  $w \in B_L$ . Take  $\tilde{w}^{g+1} = w^{g+1}, \dots, \tilde{w}^g = w^g$ . Consider then the nonlinear differential equation (NL) with initial conditions  $\tilde{w}^{(j)}(0) = w^{(j)}(0)$ , for  $0 \leq j \leq L_i - 1$  and  $1 \leq i \leq g$ . In order to associate a  $\tilde{w} \in B_{NL}$  with a  $w \in B_L$ , interchange the roles of  $B_L$  and  $B_{NL}$  in this construction. Now prove, using standard facts from the theory of differential equations, that this association satisfies the required linearization condition.

Consider, for instance, Example 1, with  $w = (u, y)$ ,  $g = 1$ ,  $q = 2$  and  $\frac{\partial f}{\partial w_2^2}(0, 0) = 1$ . Therefore  $L_1 = 2$  and the linearization is, then,  $\Delta_y^{(2)} + \Delta_y = \Delta_u$ . Now consider Example 2, with  $w = (u, y)$ ,  $\frac{\partial f}{\partial w_1^2}(0, 0) = 1$ , so that  $L_1 = 1$ . The linearization is then  $\Delta_y^{(1)} + \Delta_y = \Delta_u^{(1)}$ . Note that here  $\frac{\partial f}{\partial w_1^1}(0, 0) = -1$ .

## 5. LATENT VARIABLES

Often mathematical models obtained from first principles will contain other variables than those which we are interested in modelling. In the case of systems described by behavioral equations which are differential equations, this will lead to a model of the type

$$f(w, w^{(1)}, \dots, w^{(L)}, a, a^{(1)}, \dots, a^{(L)}) = 0. \quad (\text{NL}')$$

Formalizing this yields a dynamical system with *latent variables*  $\Sigma^l := (\mathbf{R}, \mathbf{R}^q, \mathbf{R}^l, B_f)$  relating the *signal trajectory*  $w : \mathbf{R} \rightarrow \mathbf{R}^q$  to the *latent variable trajectory*  $a : \mathbf{R} \rightarrow \mathbf{R}^l$ , via the *full behavior*  $B_f \subseteq (\mathbf{R} \times \mathbf{R}^l)^{\mathbf{R}}$ . If  $B_f$  is described by a behavioral differential equation as above,

then  $B_f$  consists of all solutions of this differential equation. Let us consider again the  $C^\infty$  case. The latent variable system  $\Sigma^l$  induces the system  $\Sigma = (\mathbf{R}, \mathbf{R}^q, B)$  with *intrinsic behavior*  $B = \{w : \mathbf{R}^q \rightarrow \mathbf{R}^q \mid \exists a : \mathbf{R} \rightarrow \mathbf{R}^l, \text{ such that } (w, a) \in B_f\}$ . If in the above differential equation  $f$  is linear, then it can be shown that  $B$  will also be described by a linear differential equation. However, for nonlinear systems this need not be the case.

The question arises if we can obtain the linearization of  $B$  from the linearization of  $B_f$ . In particular, assume that  $(w^*, a^*) \in \mathbf{R}^q \times \mathbf{R}^l$  defines an equilibrium of (NL') (assume  $w^* = 0$  and  $a^* = 0$ ). Linearizing (NL') around this equilibrium yield the linear differential equation

$$f'_{w_0} \Delta w + f'_{N_1} \Delta w^{(1)} + \dots + f'_{w_L} \Delta w^{(L)} + f'_{a_0} \Delta a + f'_{a_1} \Delta a^{(1)} + \dots + f'_{a_L} \Delta a^{(L)} = 0 \quad (L')$$

(the notation is—hopefully—self explaining). By introducing the polynomial matrices  $R(s) = f'_{w_0} + f'_{w_1} s + \dots + f'_{w_L} s^L$  and  $M(s) = f'_{a_0} + f'_{a_1} s + \dots + f'_{a_L} s^L$ , we can write this as

$$R\left(\frac{d}{dt}\right)w + M\left(\frac{d}{dt}\right)a = 0.$$

This equation describes, under conditions of the type spelled out in Section 4, the full behavior of the linearization of the latent variable system (NL'). It is well-known [4] that the intrinsic behavior of (L') can be expressed by a differential equation of the type

$$R'\left(\frac{d}{dt}\right)w = 0, \quad (L'')$$

for some suitably chosen polynomial matrix  $R'(s)$ . The question arises if this differential equation describes the linearization of the intrinsic behavior  $B$  associated with the nonlinear system (NL').

It turns out that this is the case if the system (L') is *observable*. Observability is defined abstractly in [4]. For the system (L'), it means that there must exist a polynomial matrix  $N(s)$  such that if  $(w, a)$  satisfies (L'), there will hold  $a = N\left(\frac{d}{dt}\right)w$ . Actually, as proven in [4], (L') will be observable *if and only if*  $\text{rank } M(\lambda) = \ell$ , for all  $\lambda \in C$ .

There holds that *if* (i) (L') is a linearization of (NL')(cf. Section 4) *and if* (ii) (L') is observable, *then* (L'') will define a linearization of the intrinsic behavior induced by (NL').

The linear state space system of the introduction is a linearization of the nonlinear one, if we consider in both cases the full behavior (with  $x$  as latent variable). The intrinsic behavior of the linear one will be a linearization of the intrinsic behavior of the nonlinear one if, in addition, the pair of matrices  $(\frac{\partial F}{\partial z}(0, 0), \frac{\partial H}{\partial z}(0, 0))$  is observable. That observability is not superfluous and this may be seen from the example  $\dot{x} = u, y = x^3$ . The linearization,  $\Delta x = \Delta u, \Delta y = 0$ , is not observable and the sequence of constant trajectories  $u = 0, y = \alpha$ , in the intrinsic behavior of the nonlinear system, can be made to converge to zero, but has no approximant in the linearized version!

## 6. DISCRETE TIME

For discrete time systems, identical results are obtained if the time axis is  $Z_+$  and the linearization condition is changed to

$$\frac{\sup_{0 \leq t \leq T} \|w_k(t) - \tilde{w}_k(t)\|}{\left( \sup_{0 \leq t \leq T} \|w_k(t)\| \right)} \xrightarrow{k \rightarrow \infty} 0.$$

For discrete systems with time axis  $Z$ , the polynomial matrix  $U(s)$  should be chosen such that  $\tilde{R}(s)$  is bilaterally row proper.

Examples:

4.  $y_{t+2} = y_{t+1}u_t$  is linearized by  $\Delta y_{t+2} = 0$ .
5.  $y_{t+1} = y_{t+1}u_t$  is linearized by  $\Delta y_{t+1} = 0$ .
6. The linearization of  $y_{t+2}^2 = y_{t+1}u_t$  is not clear from our results.

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